Central extensions of Steinberg Lie superalgebras of small rank

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Abstract

It was shown by A.V.Mikhalev and I.A.Pinchuk in [MP] that the second homology group $H_2(\mathfrak{st}(m,n,R))$ of the Steinberg Lie superalgebra $\mathfrak{st}(m,n,R)$ is trivial for $m+n \geq 5$. In this paper, we will work out $H_2(\mathfrak{st}(m,n,R))$ explicitly for $m+n=3,4$.

Introduction

Steinberg Lie algebras $\mathfrak{st}_n(R)$ play an important role in (additive) algebraic K-theory. They have been studied by many people (see [L] and [GS], and the references therein). The point is that for any unital associative algebra R over a field the Steinberg Lie algebra $\mathfrak{st}_n(R)$ is the universal central extension of $sl_n(R)$ with the kernel isomorphic to the first cyclic homology group $HC_1(R)$ except when both n and the characteristic of the field are small. As seen in [GS], if $n=3,4$, $H_2(\mathfrak{st}_n(R))$ is not necessarily equal to 0.

Recently, A.V.Mikhalev and I.A.Pinchuk [MP] studied the Steinberg Lie superalgebras $\mathfrak{st}(m,n,R)$ which are central extensions of Lie superalgebras $sl(m,n,R)$. They further showed that when $m+n \geq 5$, $\mathfrak{st}(m,n,R)$ is the universal central extension of $sl(m,n,R)$ whose kernel is isomorphic to $(HC_1(R))_{\bar{0}} \oplus (0)_{\bar{1}}$, here we would like to emphasize the $\mathbb{Z}_2$-gradation of the kernel.

In this paper, we shall work out $H_2(\mathfrak{st}(m,n,R))$ explicitly for $m+n=3,4$ without any assumption on char K by adopting the definition for Lie superalgebras (including char $K=2$ case) introduced by Neher [N]. It is equivalent to work on the Steinberg Lie superalgebras $\mathfrak{st}(m,n,R)$ for small $m+n$. This completes the determination of the universal central extensions of the Lie superalgebras $\mathfrak{st}(m,n,R)$ and $sl(m,n,R)$ as well.

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For any non-negative integer m, set

$$\mathcal{I}_m = mR + R[R, R] \quad \text{and} \quad R_m = R/\mathcal{I}_m.$$

Our main result of this paper is the following.

Main Theorem *Let K be a unital commutative ring and R be a unital associative K-algebra. Assume that R has a K-basis containing the identity element. Then*

$$H_2(\mathfrak{st}(2, 1, R)) = (0);$$

$$H_2(\mathfrak{st}(3, 1, R)) = (0)_{\bar{0}} \oplus (R_2^6)_{\bar{1}};$$

$$H_2(\mathfrak{st}(2, 2, R)) = (R_2^4 \oplus R_0^2)_{\bar{0}} \oplus (0)_{\bar{1}}$$

where R_m^q is the direct sum of q copies of R_m.

It may be noteworthy to point out that $H_2(\mathfrak{st}(2, 1, R)) = (0)$ unlike the Lie algebra case in which $H_2(\mathfrak{st}_3(\mathfrak{R}))$ is not necessarily zero.

The organization of this paper is as follows. In Section 1, we review some basic facts on Steinberg Lie superalgebras $\mathfrak{st}(m, n, R)$. Section 2 will treat the $m = 2, n = 1$ case. Section 3 and 4 will handle the $m = 3, n = 1$ case and the $m = 2, n = 2$ case respectively. Finally in Section 5 we make a few concluding remarks.

§1 Basics on $\mathfrak{st}(m, n, R)$

Let K be a unital commutative ring. The following definition was given in [N].

Definition A K-superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with product $[,]$ is a Lie superalgebra if for any homogenous $x, y, z \in L, w \in L_{\bar{0}}$,

$$[y, x] = -(-1)^{deg(x)deg(y)}[y, x] \tag{S1}$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{deg(x)deg(y)}[y, [x, z]] \tag{S2}$$

$$[w, w] = 0. \tag{S3}$$

Note that (S3) is not needed if char $K \neq 2$ and this definition works well for K of any characteristic.

Let R be a unital associative K-algebra. We always assume that R has a K-basis $\{r_\lambda\}_{\lambda\in\Lambda}$($\Lambda$ is an index set), which contains the identity element 1 of R, i.e. $1 \in \{r_\lambda\}_{\lambda\in\Lambda}$.

$\Omega = \{1, \ldots, m, m+1, \ldots, m+n\}$ has a partition $\Omega = \Omega_0 \uplus \Omega_1$, where $\Omega_0 = \{1, \ldots, m\}$ and $\Omega_1 = \{m+1, \ldots, m+n\}$. We define a map $\omega : \Omega \to \mathbb{Z}_2$, such that

$$\omega(i) = \begin{cases} \bar{0} & \text{for } i \in \Omega_0 \\ \bar{1} & \text{for } i \in \Omega_1 \end{cases}$$

The K-Lie superalgebra of $(m+n)\times(m+n)$ matrices with coefficients in R is denoted by $gl(m, n, R)$, such that $deg(e_{ij}(a)) = \omega(i) + \omega(j)$ for $a \in R$, $1 \le i, j \le m+n$. For $m+n \ge 3$, the elementary Lie superalgebra $sl(m, n, R)$ is the subalgebra of $gl(m, n, R)$ generated by the elements $e_{ij}(a)$, $1 \le i \ne j \le m+n$. Note that $sl(m, n, R)$ can be equivalently defined as $sl(m, n, R) = [gl(m, n, R), gl(m, n, R)]$, the derived subalgebra of $gl(m, n, R)$, or $sl(m, n, R) = \{X \in gl(m, n, R) | \operatorname{str}(X) \in [R, R]\}$, where $\operatorname{str}(X)$ is the supertrace of $X = (x_{ij}) \in M_{m+n}(R)$ given by $\operatorname{str}(X) = \sum_{i=1}^{m} x_{ii} - \sum_{j=m+1}^{m+n} x_{jj}$.

Clearly, for any $a, b \in R$,

$$[e_{ij}(a), e_{jk}(b)] = e_{ik}(ab) \tag{1.1}$$

if i, j, k are distinct and

$$[e_{ij}(a), e_{kl}(b)] = 0 \tag{1.2}$$

if $j \ne k, i \ne l$.

For $m+n \ge 3$, the Steinberg Lie superalgebra $\mathfrak{st}(m, n, R)$ is defined to be the Lie superalgebra over K generated by the homogeneous elements $X_{ij}(a)$, with $\deg(X_{ij}(a)) = \omega(i) + \omega(j)$ for any $a \in R$, $1 \le i \ne j \le m+n$, subject to the relations(see [MP]):

$$a \mapsto X_{ij}(a) \text{ is a } K\text{-linear map,} \tag{1.3}$$

$$[X_{ij}(a), X_{jk}(b)] = X_{ik}(ab), \text{ for distinct } i, j, k, \tag{1.4}$$

$$[X_{ij}(a), X_{kl}(b)] = 0, \text{ for } j \ne k, i \ne l, \tag{1.5}$$

where $a, b \in R$, $1 \le i, j, k, l \le m+n$.

Both Lie superalgebras $sl(m,n,R)$ and $\mathfrak{st}(m,n,R)$ are perfect(a Lie superalgebra $\mathfrak{g}$ over K is called perfect if $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$). The Lie superalgebra epimorphism:

$$\phi : \mathfrak{st}(m,n,R) \to sl(m,n,R), \tag{1.6}$$

such that $\phi(X_{ij}(a)) = e_{ij}(a)$, is a central extension and the kernel of ϕ is isomorphic to $HC_1(R)$ (called $HC_2(R)$ in [MP]), which is the first cyclic homology group of R. Eventually, $HC_1(R)$ is the even part of $\ker(\phi)$ and the odd part is equal to 0. So the universal central extension of $sl(m,n,R)$ is also the universal central extension of $\mathfrak{st}(m,n,R)$ denoted by $\widehat{\mathfrak{st}}(m,n,R)$. Our purpose is to calculate $\widehat{\mathfrak{st}}(m,n,R)$ for any ring K and $m+n\geq 3$.

Setting

$$T_{ij}(a,b) = [X_{ij}(a), X_{ji}(b)], \tag{1.7}$$

$$t(a,b) = T_{1j}(a,b) - T_{1j}(1,ba), \tag{1.8}$$

for $a,b\in R, 1\leq i\neq j\leq m+n$. Both $T_{ij}(a,b)$ and $t(a,b)$ are even elements. Then $t(a,b)$ does not depend on the choices of j(see [MP]). Note that $T_{ij}(a,b)$ is K-bilinear, and so is $t(a,b)$.

Lemma 1.9 *For any $a,b,c\in R$, and distinct i,j,k, we have*

$$T_{ij}(a,b) = -(-1)^{\omega(i)+\omega(j)}T_{ji}(b,a) \tag{1.10}$$

$$[T_{ij}(a,b), X_{kl}(c)] = 0 \text{ for distinct } i,j,k,l \tag{1.11}$$

$$[T_{ij}(a,b), X_{ik}(c)] = X_{ik}(abc), \quad [T_{ij}(a,b), X_{ki}(c)] = -X_{ki}(cab) \tag{1.12}$$

$$[T_{ij}(a,b), X_{jk}(c)] = -(-1)^{\omega(i)+\omega(j)}X_{jk}(bac), \quad [T_{ij}(a,b), X_{kj}(c)] = (-1)^{\omega(i)+\omega(j)}X_{kj}(cba) \tag{1.13}$$

$$[T_{ij}(a,b), X_{ij}(c)] = X_{ij}(abc + (-1)^{\omega(i)+\omega(j)}cba) \tag{1.14}$$

$$[t(a,b), X_{1i}(c)] = X_{1i}((ab-ba)c), \quad [t(a,b), X_{i1}(c)] = -X_{i1}(c(ab-ba)) \tag{1.15}$$

$$[t(a,b), X_{jk}(c)] = 0 \text{ for } j,k\geq 2 \tag{1.16}$$

Proof: By super-antisymmetry, one has:

$$T_{ij}(a,b) = -(-1)^{\omega(i)+\omega(j)}[X_{ji}(b), X_{ij}(a)] = -(-1)^{\omega(i)+\omega(j)}T_{ji}(b,a)$$

From the super-Jacobi identity, we have

$$[x,[y,z]] = [[x,y],z] + (-1)^{\deg(x)\deg(y)}[y,[x,z]]$$

which is equivalent to

$$[[x,y],z] = [x,[y,z]] + (-1)^{\deg(y)\deg(z)}[[x,z],y].$$

So (1.11) is obvious, and

$$\begin{aligned}
[T_{ij}(a,b), X_{ik}(c)] &= [[X_{ij}(a), X_{ji}(b)], X_{ik}(c)] = [X_{ij}(a), [X_{ji}(b), X_{ik}(c)]] = X_{ik}(abc) \\
[T_{ij}(a,b), X_{ki}(c)] &= [[X_{ij}(a), X_{ji}(b)], X_{ki}(c)] = (-1)^{(\omega(i)+\omega(j))(\omega(k)+\omega(i))}[[X_{ij}(a), X_{ki}(c)], X_{ji}(b)] \\
&= -(-1)^{2(\omega(i)+\omega(j))(\omega(k)+\omega(i))}[[X_{ki}(c), X_{ij}(a)], X_{ji}(b)] = -X_{ki}(cab)
\end{aligned}$$

which gives (1.12).

Replaced $T_{ij}(a,b)$ by $-(-1)^{\omega(i)+\omega(j)}T_{ji}(b,a)$ and exchanging i and j, we can obtain (1.13) from (1.12).

For (1.14), we have

$$\begin{aligned}
[T_{ij}(a,b), X_{ij}(c)] &= [T_{ij}(a,b), [X_{ik}(c), X_{kj}(1)]] \\
&= [[T_{ij}(a,b), X_{ik}(c)], X_{kj}(1)] + [X_{ik}(c), [T_{ij}(a,b), X_{kj}(1)]] \\
&= [X_{ik}(abc), X_{kj}(1)] + (-1)^{\omega(i)+\omega(j)}[X_{ik}(c), X_{kj}(ba)] \\
&= X_{ij}(abc + (-1)^{\omega(i)+\omega(j)}cba).
\end{aligned}$$

From (1.8) we obtain

$$\begin{aligned}
[t(a,b), X_{1i}(c)] &= [T_{1j}(a,b), X_{1i}(c)] - [T_{1j}(1,ba), X_{1i}(c)] \\
&= X_{1i}(abc) - X_{1i}(bac) = X_{1i}((ab-ba)c)
\end{aligned}$$

and $[t(a,b), X_{i1}(c)] = -X_{i1}(c(ab-ba))$, which show that (1.15) holds true.

(1.16) is easy and the proof is completed. $\square$

By the above Lemma, we have

Lemma 1.17 *Let* $\mathfrak{T} := \sum_{1\le i<j\le m+n}[X_{ij}(R), X_{ji}(R)]$. *Then* $\mathfrak{T}$ *is a subalgebra of* $\mathfrak{st}(m,n,R)$ *containing the center* $\mathfrak{Z}$ *of* $\mathfrak{st}(m,n,R)$ *with* $[\mathfrak{T}, X_{ij}(R)] \subseteq X_{ij}(R)$. *Moreover,*

$$\mathfrak{st}(m,n,R) = \mathfrak{T} \oplus_{1\le i\ne j\le m+n} X_{ij}(R). \tag{1.18}$$

As for the decomposition of $\mathfrak{st}(m,n,R)$, we take $\{r_\lambda\}_{\lambda\in\Lambda}$, the fixed K-basis of R, then $\{X_{ij}(r)\}$ $(r \in \{r_\lambda\}_{\lambda\in\Lambda}, 1 \le i \ne j \le m+n)$ can be extended to a K-basis Γ of $\mathfrak{st}(m,n,R)$.

In fact, the subalgebra $\mathfrak{T}$ has a more refined structure.

One can easily prove the following lemma (see [MP]).

Lemma 1.19 *Every element* $x \in \mathfrak{T}$ *can be written as*

$$x = \sum_i t(a_i, b_i) + \sum_{2\le j\le m+n} T_{1j}(1, c_j),$$

where $a_i, b_i, c_j \in R$.

The following result is known(see [MP, Theorem 2]).

Theorem 1.20 *If* $m+n \ge 5$, *then* $\phi : \mathfrak{st}(m,n,R) \to sl(m,n,R)$ *gives the universal central extension of* $sl(m,n,R)$ *and the second homology group of Lie superalgebra* $\mathfrak{st}(m,n,R)$ *is* $H_2(\mathfrak{st}(m,n,R)) = 0$.

§2 Central extensions of $\mathfrak{st}(2,1,R)$

In this section we shall treat $H_2(\mathfrak{st}(2,1,R))$.

Theorem 2.1 $H_2(\mathfrak{st}(2,1,R)) = 0$, *i.e.* $\mathfrak{st}(2,1,R)$ *is centrally closed.*

Proof: Suppose that

$$0 \to \mathcal{V} \to \widetilde{\mathfrak{st}}(2,1,R) \xrightarrow{\tau} \mathfrak{st}(2,1,R) \to 0$$

is a central extension of $\mathfrak{st}(2,1,R)$. We must show that there exists a Lie superalgebra homomorphism $\eta : \mathfrak{st}(2,1,R) \to \widetilde{\mathfrak{st}}(2,1,R)$ so that $\tau \circ \eta = \mathrm{id}$.

Using the K-basis $\{r_\lambda\}_{\lambda\in\Lambda}$ of R, we choose a preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ under τ, $1 \le i \ne j \le 3, a \in \{r_\lambda\}_{\lambda\in\Lambda}$. Let $\widetilde{T}_{ij}(a,b) = [\widetilde{X}_{ij}(a), \widetilde{X}_{ji}(b)]$, then

$$[\widetilde{T}_{ik}(1,1), \widetilde{X}_{ij}(a)] = \widetilde{X}_{ij}(a) + \mu_{ij}(a)$$

where $\mu_{ij}(a) \in \mathcal{V}$ and i, j, k are distinct. Replacing $\widetilde{X}_{ij}(a)$ by $\widetilde{X}_{ij}(a) + \mu_{ij}(a)$, then the elements $\widetilde{X}_{ij}(b)$ still satisfy the relations (1.3). By super-Jacobi identity (S2), we have

$$\Big[\widetilde{T}_{ik}(1,1), [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)]\Big] = \Big[[\widetilde{T}_{ik}(1,1), \widetilde{X}_{ik}(a)], \widetilde{X}_{kj}(b)\Big] + \Big[\widetilde{X}_{ik}(a), [\widetilde{T}_{ik}(1,1), \widetilde{X}_{kj}(b)]\Big]$$

which yields

$$\begin{aligned}[\widetilde{T}_{ik}(1,1), \widetilde{X}_{ij}(ab)] &= [\widetilde{X}_{ik}(a + (-1)^{\omega(i)+\omega(k)} a), \widetilde{X}_{kj}(b)] - (-1)^{\omega(i)+\omega(k)} [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] \\ &= [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)].\end{aligned}$$

We thus have

$$\widetilde{X}_{ij}(ab) = [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)]. \tag{2.2}$$

For $k \neq i, k \neq j$, we have

$$\begin{aligned}&[\widetilde{X}_{ij}(a), \widetilde{X}_{ij}(b)] &(1)\\ =&\Big[\widetilde{X}_{ij}(a), [\widetilde{X}_{ik}(b), \widetilde{X}_{kj}(1)]\Big] \\ =&\Big[[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)], \widetilde{X}_{kj}(1)\Big] + \Big[\widetilde{X}_{ik}(b), (-1)^{(\omega(i)+\omega(j))(\omega(i)+\omega(k))} [\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(1)]\Big] \\ =&0 + 0 = 0. &(2.3)\end{aligned}$$

Next, we show that both of $[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)]$ and $[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(b)]$ are equal to 0.

Since there is always one element between $\widetilde{X}_{ij}(a)$ and $\widetilde{X}_{ik}(b)$ which is odd, we can assume that it is $\widetilde{X}_{ij}(a)$, i.e. $\omega(i) + \omega(j) = \bar{1}$, then

$$\begin{aligned}0 &= \Big[\widetilde{T}_{ij}(1,1), [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)]\Big] &(2)\\ &= \Big[[\widetilde{T}_{ij}(1,1), \widetilde{X}_{ij}(a)], \widetilde{X}_{ik}(b)\Big] + \Big[\widetilde{X}_{ij}(a), [\widetilde{T}_{ij}(1,1), \widetilde{X}_{ik}(b)]\Big] \\ &= 0 + [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] = [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] &(2.4)\end{aligned}$$

The other cases are similar. Therefore we have

$$[X_{ij}(a), X_{kl}(b)] = 0, \text{ for } j \neq k, i \neq l, a, b \in R, 1 \leq i, j, k, l \leq 3. \tag{2.5}$$

By our choices, we know that $\widetilde{X}_{ij}(a)$ satisfy the relation (1.3)-(1.5). Since we have (2.2) and (2.5), by universal property of $\mathfrak{st}(2,1,R)$ there exists a (unique) Lie superalgebra homomorphism

$$\eta : \mathfrak{st}(2,1,R) \to \widetilde{\mathfrak{st}}(2,1,R)$$

such that $\eta(X_{ij}(a)) = \widetilde{X}_{ij}(a)$. Evidently, $\tau \circ \eta = id$ which implies that the original sequence splits. So $\mathfrak{st}(2,1,R)$ is centrally closed. $\square$

Remark 2.6 This result is very different from the one of $\mathfrak{st}_3(R)$(See [GS]). In that case, $H_2(\mathfrak{st}_3(R)) = R_3^6$.

§3 Central extensions of $\mathfrak{st}(3,1,R)$

In this section, we shall compute the universal central extension $\widehat{\mathfrak{st}}(3,1,R)$ of $\mathfrak{st}(3,1,R)$. We don't put any assumption on the characteristic of K.

For any nonnegative integer m, let $\mathcal{I}_m$ be the ideal of R generated by the elements: ma and $ab - ba$, for $a, b \in R$. Immediately, we have ([GS, Lemma 2.1])

Lemma 3.1 *$\mathcal{I}_m = mR + R[R,R]$ and $[R,R]R = [R,R]R$.*

Let

$$R_m := R/\mathcal{I}_m$$

be the quotient algebra over K which is commutative. Write $\bar{a} = a + \mathcal{I}_m$ for $a \in R$. Note that if $m = 2$, $\overline{a} = -\overline{a}$ in R_m.

Definition 3.2 *$\mathcal{W} = R_2^6$ is the direct sum of six copies of R_2 and $\epsilon_m(\overline{a}) = (0, \cdots, \overline{a}, \cdots, 0)$ is the element of $\mathcal{W}$, of which the m-th component is $\overline{a}$ and others are zero, for $1 \leq m \leq 6$.*

Let S_4 be the symmetric group of $\{1,2,3,4\}$.

$$P = \{(i,j,k,l) | \{i,j,k,l\} = \{1,2,3,4\}\}$$

is the set of all the quadruple with the distinct components. S_4 has a natural transitive action on P given by $\sigma((i,j,k,l)) = (\sigma(i), \sigma(j), \sigma(k), \sigma(l))$, for any $\sigma \in S_4$.

$$H = \{(1), (13), (24), (13)(24)\}$$

is a subgroup of S_4 with $[S_4 : H] = 6$. Then S_4 has a partition of cosets with respect to H, denoted by $S_4 = \bigsqcup_{m=1}^{6} \sigma_m H$. We can obtain a partition of P, $P = \bigsqcup_{m=1}^{6} P_m$, where $P_m = (\sigma_m H)((1,2,3,4))$. We define the index map

$$\theta : P \to \{1,2,3,4,5,6\}$$

by

$$\theta\left((i,j,k,l)\right) = m \text{ if } (i,j,k,l) \in P_m,$$

for $1 \leq m \leq 6$.

Using the decomposition (1.18) of $\mathfrak{st}(3,1,R)$, we take a K-basis Γ of $\mathfrak{st}(3,1,R)$, which contains $\{X_{ij}(r) | r \in \{r_\lambda\}_{\lambda\in\Lambda}, 1 \leq i \neq j \leq 4\}$. Define $\psi : \Gamma \times \Gamma \to \mathcal{W}$ by

$$\psi(X_{ij}(r), X_{kl}(s)) = \epsilon_{\theta((i,j,k,l))}(\overline{rs}) \in \mathcal{W},$$

for $r, s \in \{r_\lambda\}_{\lambda\in\Lambda}$ and distinct i, j, k, l and $\psi = 0$, otherwise. Then we obtain the K-bilinear map $\psi : \mathfrak{st}(3,1,R) \times \mathfrak{st}(3,1,R) \to \mathcal{W}$ by linearity.

Recall that a Lie superalgebra over K is defined to be an $\mathbb{Z}_2$-graded algebra satisfying $[x, y] = -(-1)^{deg(x)deg(y)}[y, x]$,

$$(-1)^{deg(x)deg(z)}[[x, y], z] + (-1)^{deg(x)deg(y)}[[y, z], x] + (-1)^{deg(y)deg(z)}[[z, x], y] = 0$$

and $[w, w] = 0$ for the homogenous elements $x, y, z \in L$ and $w \in L_{\bar{0}}$.

We now have

Lemma 3.3 *The bilinear map ψ is a (super) 2-cocycle.*

Proof: A bilinear map ψ is called a (super) 2-cocycle, if it is (super) skew-symmetric and

$$(-1)^{deg(x)deg(z)}\psi([x, y], z) + (-1)^{deg(x)deg(y)}\psi([y, z], x) + (-1)^{deg(y)deg(z)}\psi([z, x], y) = 0$$

for homogenous elements $x, y, z \in L$ and $\psi(w, w) = 0$ for $w \in L_{\bar{0}}$.

Since $R_2 = R/\mathcal{I}_2$, $\overline{ab} = \overline{a}\overline{b} = \overline{b}\overline{a} = \overline{ba}$ and $\overline{a} = -\overline{a}$ for $a, b \in R$. Thus the order of factors and $\pm$ sign don't play any role. We can follow the same arguments as in [GS, Lemma 2.3] for Steinberg Lie algebra $st_4(R)$ to complete the proof. $\Box$

Since

$$\mathcal{W} = span_K\{\psi(X_{ij}(a), X_{kl}(b)) | a, b \in R \text{ and } i, j, k, l \text{ are distinct}\}$$

and

$$\omega(i) + \omega(j) + \omega(k) + \omega(l) = \bar{1} \text{ for distinct } i, j, k, l,$$

we obtain a central extension of Lie superalgebra $\mathfrak{st}(3,1,R)$, satisfying that $\mathcal{W}$ is the odd part of the kernel :

$$0 \to (0)_{\bar{0}} \oplus (\mathcal{W})_{\bar{1}} \to \widehat{\mathfrak{st}}(3,1,R) \xrightarrow{\pi} \mathfrak{st}(3,1,R) \to 0, \tag{3.4}$$

i.e.

$$\widehat{\mathfrak{st}}(3,1,R) = ((0)_{\bar{0}} \oplus (\mathcal{W})_{\bar{1}}) \oplus \mathfrak{st}(3,1,R), \tag{3.5}$$

with bracket

$$[(c,x),(c',y)] = (\psi(x,y),[x,y])$$

for all $x, y \in \mathfrak{st}(3,1,R)$ and $c, c' \in \mathcal{W}$, where $\pi : \mathcal{W} \oplus \mathfrak{st}(3,1,R) \to \mathfrak{st}(3,1,R)$ is the second coordinate projection map. Then, $(\widehat{\mathfrak{st}}(3,1,R),\pi)$ is a central extension of $\mathfrak{st}(3,1,R)$. We will show that $(\widehat{\mathfrak{st}}(3,1,R),\pi)$ is the universal central extension of $\mathfrak{st}(3,1,R)$. To do this, we define a Lie superalgebra $\mathfrak{st}(3,1,R)^\sharp$ to be the Lie superalgebra generated by the symbols $X^\sharp_{ij}(a)$, $i \neq j, a \in R$ and the K-linear space $\mathcal{W}$, with $\deg(X^\sharp_{ij}(a)) = \omega(i) + \omega(j)$ and $\deg(w) = \bar{1}$ for any $w \in \mathcal{W}$, satisfying the following relations:

$$a \mapsto {X_{ij}}^\sharp(a) \text{ is a } K\text{-linear mapping}, \tag{3.6}$$

$$[X^\sharp_{ij}(a), X^\sharp_{jk}(b)] = X^\sharp_{ik}(ab), \text{ for distinct } i,j,k, \tag{3.7}$$

$$[X^\sharp_{ij}(a), \mathcal{W}] = 0, \text{ for distinct } i,j, \tag{3.8}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ij}(b)] = 0, \text{ for distinct } i,j, \tag{3.9}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ik}(b)] = 0, \text{ for distinct } i,j,k, \tag{3.10}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kj}(b)] = 0, \text{ for distinct } i,j,k, \tag{3.11}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kl}(b)] = \epsilon_{\theta((i,j,k,l))}(\overline{ab}), \text{ for distinct } j,k,i,l, \tag{3.12}$$

where $a, b \in R$,$1 \leq i,j,k,l \leq 4$. As $1 \in R$, $\mathfrak{st}(3,1,R)^\sharp$ is perfect. Clearly, there is a unique Lie superalgebra homomorphism $\rho : \mathfrak{st}(3,1,R)^\sharp \to \widehat{\mathfrak{st}}(3,1,R)$ such that $\rho(X^\sharp_{ij}(a)) = X_{ij}(a)$ and $\rho|_{\mathcal{W}} = id$.

Remark 3.13: Comparing with the relations of $\mathfrak{st}(m,n,R)$ (1.3)-(1.5), we separate the case $[X^\sharp_{ij}(a), X^\sharp_{kl}(b)](j \neq k, i \neq l)$ into four subcases (3.9)-(3.12).

We claim that ρ is actually an isomorphism.

Lemma 3.14 $\rho : \mathfrak{st}(3,1,R)^\sharp \to \widehat{\mathfrak{st}}(3,1,R)$ *is a Lie superalgebra isomorphism.*

Proof: Let $T^\sharp_{ij}(a,b) = [X^\sharp_{ij}(a), X^\sharp_{ji}(b)]$. Then one can easily check that for $a, b \in R$ and distinct i, j, k, one has

$$T^\sharp_{ij}(a,b) = -(-1)^{\omega(i)+\omega(j)} T^\sharp_{ji}(b,a) \tag{3.15}$$

$$T^\sharp_{ij}(ab,c) = T^\sharp_{ik}(a,bc) + (-1)^{\omega(i)+\omega(k)} T^\sharp_{kj}(b,ca). \tag{3.16}$$

Indeed, the proof of (3.16) is the same as the proof in [MP, Lemma 4.1]. Put $t^\sharp(a,b) = T^\sharp_{1j}(a,b) - T^\sharp_{1j}(1,ab)$ for $a, b \in R, 2 \le j \le 4$. Then $t^\sharp(a,b)$ does not depend on the choice of j. Also, one can easily check (as in the proof of Lemma 2.15 in [GS]) that

$$\mathfrak{st}(3,1,R)^\sharp = \mathfrak{T}^\sharp \oplus_{1\le i\ne j\le 4} X^\sharp_{ij}(R)$$

where

$$\mathfrak{T}^\sharp = \left(\sum_{i,j,k,l \text{ are distinct}} [X^\sharp_{ij}(R), X^\sharp_{kl}(R)] \right) \oplus \left(\sum_{1\le i<j\le 4} [X^\sharp_{ij}(R), X^\sharp_{ji}(R)] \right).$$

It then follows from (3.15) and (3.16) above that

$$\mathfrak{T}^\sharp = \mathcal{W} \oplus \Big(t^\sharp(R,R) \oplus T^\sharp_{12}(1,R) \oplus T^\sharp_{13}(1,R) \oplus T^\sharp_{14}(1,R) \Big) \tag{3.17}$$

where $t^\sharp(R,R)$ is the linear span of the elements $t^\sharp(a,b)$. So by Lemma 1.19, it suffices to show that the restriction of ρ to $t^\sharp(R,R)$ is injective.

Now the similar argument as given in [AG, Lemma 6.18] shows that there exists a linear map from $t(R,R)$ to $t^\sharp(R,R)$ so that $t(a,b) \mapsto t^\sharp(a,b)$ for $a, b \in R$. This map is the inverse of the restriction of ρ to $t^\sharp(R,R)$. $\Box$

The following theorem is the main result of this section:

Theorem 3.18 $(\widehat{\mathfrak{st}}(3,1,R), \pi)$ *is the universal central extension of* $\mathfrak{st}(3,1,R)$ *and hence*

$$H_2(\mathfrak{st}(3,1,R)) \cong (0)_{\bar{0}} \oplus (\mathcal{W})_{\bar{1}}.$$

Proof: We imitate the method of proving the universal central extension of $\mathfrak{st}_4(R)$ in [GS].

Suppose that

$$0 \to \mathcal{V} \to \widetilde{\mathfrak{st}}(3,1,R) \xrightarrow{\tau} \mathfrak{st}(3,1,R) \to 0$$

is a central extension of $\mathfrak{st}(3,1,R)$. We must show that there exists a Lie superalgebra homomorphism $\eta : \widehat{\mathfrak{st}}(3,1,R) \to \widetilde{\mathfrak{st}}(3,1,R)$ so that $\tau \circ \eta = \pi$. Thus, by Lemma 3.14, it suffices to show that there exists a Lie superalgebra homomorphism $\xi : \mathfrak{st}(3,1,R)^\sharp \to \widetilde{\mathfrak{st}}(3,1,R)$ such that $\tau \circ \xi = \pi \circ \rho$.

Using the K-basis $\{r_\lambda\}_{\lambda\in\Lambda}$ of R, we choose a preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ under τ, $1 \le i \ne j \le 4, a \in \{r_\lambda\}_{\lambda\in\Lambda}$, so that the elements $\widetilde{X}_{ij}(a)$ satisfy the relations (3.6)-(3.12). For distinct i, j, k, let

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab) + \mu^k_{ij}(a, b)$$

where $\mu^k_{ij}(a, b) \in \mathcal{V}$. Take distinct i, j, k, l, then

$$\Big[\widetilde{X}_{ik}(a), [\widetilde{X}_{kl}(c), \widetilde{X}_{lj}(b)]\Big] = [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(cb)].$$

But the left side is, by super-Jacobi identity,

$$\Big[[\widetilde{X}_{ik}(a), \widetilde{X}_{kl}(c)], \widetilde{X}_{lj}(b)\Big] + (-1)^{(\omega(i)+\omega(k))(\omega(k)+\omega(l))}\Big[\widetilde{X}_{kl}(c), [\widetilde{X}_{ik}(a), \widetilde{X}_{lj}(b)]\Big] = [\widetilde{X}_{il}(ac), \widetilde{X}_{lj}(b)].$$

as $[\widetilde{X}_{ik}(a), \widetilde{X}_{lj}(b)] \in \mathcal{V}$. Thus

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(cb)] = [\widetilde{X}_{il}(ac), \widetilde{X}_{lj}(b)].$$

In particular, $\mu^k_{ij}(a, cb) = \mu^l_{ij}(ac, b)$ and $[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = [\widetilde{X}_{il}(a), \widetilde{X}_{lj}(b)]$. It follows that $\mu^k_{ij}(a, b) = \mu^l_{ij}(a, b) = \mu_{ij}(a, b)$ which show $\mu^k_{ij}(a, b)$ is independent of the choice of k and $\mu_{ij}(c, b) = \mu_{ij}(1, cb)$, we have

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab) + \mu_{ij}(a, b).$$

Taking $a = 1$, we have

$$[\widetilde{X}_{ik}(1), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(b) + \mu_{ij}(1, b).$$

Now, we replace $\widetilde{X}_{ij}(b)$ by $\widetilde{X}_{ij}(b) + \mu_{ij}(1, b)$. Then the elements $\widetilde{X}_{ij}(b)$ still satisfy the relations (3.6). Moreover we have

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab) \tag{3.19}$$

for $a, b \in R$ and distinct i, j, k. So the elements $\widetilde{X}_{ij}(a)$ satisfy (3.7).

Next for $k \neq i, k \neq j$, we have

$$\begin{aligned}
&[\widetilde{X}_{ij}(a), \widetilde{X}_{ij}(b)] &(3)\\
=&\left[\widetilde{X}_{ij}(a), [\widetilde{X}_{ik}(b), \widetilde{X}_{kj}(1)]\right]\\
=&\left[[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)], \widetilde{X}_{kj}(1)\right] + (-1)^{(\omega(i)+\omega(j))(\omega(i)+\omega(k))}\left[\widetilde{X}_{ik}(b), [\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(1)]\right]\\
=&0+0=0 &(3.20)
\end{aligned}$$

as both $[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)]$ and $[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(1)]$ are in $\mathcal{V}$. Thus, we get the relation (3.9).

For (3.10), taking $l \notin \{i, j, k\}$

$$\begin{aligned}
&[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] &(4)\\
=&\left[\widetilde{X}_{ij}(a), [\widetilde{X}_{il}(b), \widetilde{X}_{lk}(1)]\right]\\
=&\left[[\widetilde{X}_{ij}(a), \widetilde{X}_{il}(b)], \widetilde{X}_{kj}(1)\right] + (-1)^{(\omega(i)+\omega(j))(\omega(i)+\omega(l))}\left[\widetilde{X}_{il}(b), [\widetilde{X}_{ij}(a), \widetilde{X}_{lk}(1)]\right]\\
=&0+0=0 &(3.21)
\end{aligned}$$

with $[\widetilde{X}_{ij}(a), \widetilde{X}_{il}(b)], [\widetilde{X}_{ij}(a), \widetilde{X}_{lk}(1)] \in \mathcal{V}$. Similarly, we have

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(b)] = 0 \tag{3.22}$$

for distinct i, j, k, which is the relation (3.11).

To verify (3.12) one needs a few more steps. First, set $\widetilde{T}_{ij}(a, b) = [\widetilde{X}_{ij}(a), \widetilde{X}_{ji}(b)]$. The following brackets are easily checked by the super-Jacobi identity.

$$\begin{aligned}
[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ik}(c)] = \widetilde{X}_{ik}(abc),\ \ [\widetilde{T}_{ij}(a, b), \widetilde{X}_{kj}(c)] = (-1)^{\omega(i)+\omega(j)}\widetilde{X}_{kj}(cba)\\
\text{and } [\widetilde{T}_{ij}(a, b), \widetilde{X}_{kl}(c)] = 0. \qquad (3.23)
\end{aligned}$$

Then we have

$$\begin{aligned}
&[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ij}(c)] &(5)\\
=&\left[\widetilde{T}_{ij}(a, b), [\widetilde{X}_{ik}(c), \widetilde{X}_{kj}(1)]\right]\\
=&\left[[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ik}(c)], \widetilde{X}_{kj}(1)\right] + \left[\widetilde{X}_{ik}(c), [\widetilde{T}_{ij}(a, b), \widetilde{X}_{kj}(1)]\right]\\
=&\widetilde{X}_{ij}(abc) + (-1)^{\omega(i)+\omega(j)}\widetilde{X}_{ij}(cba) &(6)\\
=&\widetilde{X}_{ij}(abc + (-1)^{\omega(i)+\omega(j)}cba) &(3.24)
\end{aligned}$$

for $a,b,c\in R$ and distinct i,j,k,l.

Next for distinct i,j,k,l, let

$$[\widetilde{X}_{ij}(a),\widetilde{X}_{kl}(b)]=\nu^{ij}_{kl}(a,b)$$

where $\nu^{ij}_{kl}(a,b)\in\mathcal{V}$.

Since one and only one between $\widetilde{X}_{ij}(a)$ and $\widetilde{X}_{kl}(b)$ is even, we can assume $\deg(\widetilde{X}_{ij}(a))=\bar{0}$. By (3.23) and (3.24),

$$\begin{aligned}2\nu^{ij}_{kl}(a,b)&=[\widetilde{X}_{ij}(2a),\widetilde{X}_{kl}(b)]=\Big[[\widetilde{T}_{ij}(1,1),\widetilde{X}_{ij}(a)],\widetilde{X}_{kl}(b)\Big]\\&=\Big[\widetilde{T}_{ij}(1,1),[\widetilde{X}_{ij}(a),\widetilde{X}_{kl}(b)]\Big]+\Big[[\widetilde{T}_{ij}(1,1),\widetilde{X}_{kl}(b)],\widetilde{X}_{ij}(a)\Big]\\&=0\end{aligned}$$

which yields

$$\nu^{ij}_{kl}(a,b)=-\nu^{ij}_{kl}(a,b). \tag{3.25}$$

for any distinct $1\le i,j,k,l\le 4$ and $a,b\in R$.

Thus, with the universal property of $\mathfrak{st}(3,1,R)^\sharp$, we can obtain the Lie superalgebra homomorphism $\xi:\mathfrak{st}(3,1,R)^\sharp\to\widetilde{\mathfrak{st}}(3,1,R)$ so that $\tau\circ\xi=\pi\circ\rho$ (as was done in the proof of [GS, Theorem 2.19]). $\square$

Remark 3.26 If 2 is an invertible element of K, then $R=2R$. Thus $\mathcal{I}_2=R$ and $\mathcal{W}=R_2^6=0$. In this case, $\mathfrak{st}(3,1,R)$ is centrally closed.

If the characteristic of K is 2, we display the following two examples which are two extreme cases.

Example 3.27 Let R be an associative commutative K-algebra where char $K=2$, then we have $\mathcal{I}_2=0$ and $R_2=R$. Therefore $H_2(\mathfrak{st}(3,1,R))=R^6$.

Example 3.28 Let K be a field of characteristic two. $R=W_k$ is the Weyl algebra which is a unital associative algebra over K generated by $x_1,\ldots,x_k,y_1,\ldots,y_k$ subject to the relations $x_ix_j=x_jx_i,\ y_iy_j=y_jy_i,\ x_iy_j-y_jx_i=\delta_{ij}$. Then $\mathcal{I}_2=R$, $H_2(\mathfrak{st}(3,1,R))=0$ and $\mathfrak{st}(3,1,R)$ is centrally closed.

§4 Central extensions of $\mathfrak{st}(2,2,R)$

In this section, we compute the universal central extension $\widehat{\mathfrak{st}}(2,2,R)$ of $\mathfrak{st}(2,2,R)$.

Definition 4.1 *$\mathcal{U} = R_2^4 \oplus R_0^2$ is the direct sum of four copies of R_2 and two copies of R_0, $\epsilon_m(\bar{a}) = (0, \cdots, \overline{a}, \cdots, 0)$ is the element of $\mathcal{U}$, of which the m-th component is $\overline{a}$ and others are zero, for $1 \leq m \leq 6$*

Recall the set of all the quadruple with the distinct components P and the action of S_4 on P. $H = \{(1), (13), (24), (13)(24)\}$ is the subgroup of S_4, and P have the partition $P = \bigsqcup_{m=1}^{6} P_m$, where $P_m = (\sigma_m H)((1,2,3,4))$(cf. Section 3).

Since the index set of $\mathfrak{st}(2,2,R)$ is $\Omega = \{1,2\} \uplus \{3,4\}$, we need to classify the subsets P_m of P in this partition.

Proposition 4.2 *If an element $(i,j,k,l) \in P_m$ satisfies $\omega(i) = \omega(k)$, then all the elements of P_m have this property.*

Proof: In fact, $\omega(i) = \omega(k)$ induces $\omega(j) = \omega(l)$. It is easy to see that it is preserved under the action of H. The result is obvious. $\square$

One can easily see that the ones of P_m satisfying the above property are:

$$\{(1,3,2,4), (1,4,2,3), (2,3,1,4), (2,4,1,3)\}$$

and

$$\{(3,1,4,2), (3,2,4,1), (4,1,3,2), (4,2,3,1)\}.$$

We denote them by P_5 and P_6 respectively. Fix the index map $\theta : P \to \{1,2,3,4,5,6\}$ satisfying $\theta((1,3,2,4)) = 5$, $\theta((3,1,4,2)) = 6$. For $(i,j,k,l) \in P_5 \bigsqcup P_6$, we always have $\omega(i) + \omega(j) = \bar{1}$ and $\omega(k) + \omega(l) = \bar{1}$.

Here we single out P_5 and P_6 from the others. This is because for $1 \leq m \leq 4$, there exist elements (i,j,k,l) of P_m such that $\omega(i) + \omega(j) = \bar{0}$, but it is not true for P_5 and P_6.

Now we take a K-basis Γ of $\mathfrak{st}(3,1,R)$, which contains $\{X_{ij}(r) | r \in \{r_\lambda\}_{\lambda \in \Lambda}, 1 \leq i \neq j \leq 4\}$. Define $\psi : \Gamma \times \Gamma \to \mathcal{W}$ by

$$\psi(X_{ij}(r), X_{kl}(s)) = sign((i,j,k,l))\epsilon_{\theta((i,j,k,l))}(\overline{rs})$$

for $r, s \in \{r_\lambda\}_{\lambda \in \Lambda}$, $(i,j,k,l) \in P$ and $\psi = 0$, otherwise.

We take the the symbols $sign((i,j,k,l)) = 1$ for $(i,j,k,l) \in \bigsqcup_{m=1}^{4} P_m$, and

$$sign((i,j,k,l)) = \begin{cases} 1 & if\ (i,j,k,l) = (1,3,2,4), (2,4,1,3), (3,1,4,2), (4,2,3,1) \\ -1 & if\ (i,j,k,l) = (1,4,2,3), (2,3,1,4), (3,2,4,1), (4,1,3,2) \end{cases}$$

on $P_5 \bigsqcup P_6$

We then have

Lemma 4.3 *The bilinear map ψ is a (super) 2-cocycle.*

Proof: By the definition, one can check the (super) skew-symmetry of ψ.

In fact, if $1 \le m \le 4$, $\epsilon_m(\bar{a}) = -\epsilon_m(\bar{a})$, thus the $\pm$ sign don't play any role for $(i,j,k,l) \in \bigsqcup_{m=1}^{4} P_m$. On the other hand,

$$\psi(X_{ij}(r), X_{kl}(s)) = \psi(X_{kl}(r), X_{ij}(s)) = -(-1)^{(\omega(i)+\omega(j))(\omega(k)+\omega(l))}\psi(X_{kl}(r), X_{ij}(s)),$$

for $(i,j,k,l) \in P_5 \bigsqcup P_6$, $\omega(i)+\omega(j) = \omega(k)+\omega(l) = \bar{1}$ and the definition of $sign((i,j,k,l))$.

Moreover, ψ is skew-symmetric on $\mathfrak{st}(2,2,R)_{\bar{0}}$ and it is clear that $\psi(\gamma,\gamma) = 0$ for γ is contained in the fixed K-basis Γ of $\mathfrak{st}(2,2,R)_{\bar{0}}$, which implies $\psi(w,w) = 0$, for any $w \in \mathfrak{st}(2,2,R)_{\bar{0}}$.

Next, we should show $J(x,y,z) = 0$, where

$$J(x,y,z) = (-1)^{deg(x)deg(z)}\psi([x,y],z) + (-1)^{deg(x)deg(y)}\psi([y,z],x) + (-1)^{deg(y)deg(z)}\psi([z,x],y)$$

for the homogenous elements x, y, z. According to Lemma 1.17 and Lemma 1.19, the Steinberg Lie superalgebra $\mathfrak{st}(2,2,R)$ has the decomposition :

$$\begin{aligned}\mathfrak{st}(2,2,R) =& t(R,R) \oplus T_{12}(1,R) \oplus T_{13}(1,R) \oplus T_{14}(1,R) \\ & \oplus_{1 \le i \ne j \le n} X_{ij}(R),\end{aligned} \tag{4.4}$$

where $t(R,R)$ is the K-linear span of the elements $t(a,b)$.

We will show the following two possibilities:

Case 1: Clearly, the number of elements of x, y, z belonging to the subalgebra $\mathfrak{T}$ such that $\psi([x,y],z) \neq 0$ is at most one. Thus we can suppose that $x = X_{ij}(a), y = X_{kl}(b)$ and

$z \in \mathfrak{T}$. If $(i,j,k,l) \in \bigsqcup_{m=1}^{4} P_m$, it is similar with the proof of [GS, Lemma2.3]. Therefore, we only should consider $(i,j,k,l) \in P_5 \bigsqcup P_6$. Fix $x = X_{13}(a), y = X_{24}(b)$ and omit the other similar cases. By (4.4), we can assume that either $z = t(c,d)$, where $c, d \in R$, or $z = T_{1j}(1,c)$, where $2 \leq j \leq 4$ and $c \in R$. Note that $\deg(z) = \bar{0}$, $\theta((1,3,2,4)) = \theta((2,4,1,3)) = 5$ and $sign((1,3,2,4)) = sign((2,4,1,3)) = 1$. By Lemma 1.9, when $z = t(a,b)$, we have

$$\begin{aligned} J(x,y,z)) &= \psi([t(c,d), X_{13}(a)], X_{24}(b)) \\ &= \psi(X_{13}((cd-dc)a), X_{24}(b)) \\ &= \epsilon_5(\overline{(cd-dc)ab}) = 0; \end{aligned}$$

when $z = T_{12}(c)$,

$$\begin{aligned} J(x,y,z) &= -\psi([X_{24}(b), T_{12}(1,c)], X_{13}(a)) + \psi([T_{12}(1,c), X_{13}(a)], X_{24}(b)) \\ &= \psi([T_{12}(1,c), X_{24}(b)], X_{13}(a)) + \psi(X_{13}(ca), X_{24}(b)) \\ &= -\psi(X_{24}(cb), X_{13}(a)) + \psi(X_{13}(ca), X_{24}(b)) \\ &= -\epsilon_5(\overline{cba}) + \epsilon_5(\overline{cab}) = \epsilon_5(\overline{c(ab-ba)}) = 0; \end{aligned}$$

when $z = T_{13}(c)$,

$$\begin{aligned} J(x,y,z) &= -\psi([X_{24}(b), T_{13}(c)], X_{13}(a)) + \psi([T_{13}(1,c), X_{13}(a)], X_{24}(b)) \\ &= 0 + \psi(X_{13}(ac + (-1)^{\omega(1)+\omega(3)}ca), X_{3,4}(b)) \\ &= \epsilon_5((\overline{ac-ca)b}) = 0; \end{aligned}$$

when $z = T_{14}(c)$,

$$\begin{aligned} J(x,y,z) &= -\psi([X_{24}(b), T_{14}(c)], X_{13}(a)) + \psi([T_{14}(1,c), X_{13}(a)], X_{2,4}(b)) \\ &= \psi([T_{14}(c), X_{24}(b)], X_{13}(a)) + \psi(X_{13}(ca), X_{24}(b)) \\ &= -\psi(X_{24}(bc), X_{13}(a)) + \psi(X_{13}(ca), X_{24}(b)) \\ &= -\epsilon_5(\overline{bca}) + \epsilon_5(\overline{cab}) = \epsilon_5(\overline{cab-bca}) = 0. \end{aligned}$$

Case 2: If there is none of $\{x,y,z\}$ belonging to $\mathfrak{T}$, the nonzero terms of $J(x,y,z)$ must be $\psi([X_{ik}(a), X_{kj}(b)], X_{kl}(c))$ or $\psi([X_{il}(a), X_{lj}(b)], X_{kl}(c))$, for distinct i,j,k,l and $a,b,c \in R$.

If $(i,j,k,l) \in \bigsqcup_{m=1}^{4} P_m$, it is the same as Case 2 in the proof of [GS, Lemma 2.3]. Thus, it is enough to check the following two subcases.

One is: $x = X_{12}(a)$,$y = X_{23}(b)$, $z = X_{24}(c)$, and

$$\begin{aligned} J(x,y,z) &= \psi(X_{13}(ab), X_{24}(c)) - \psi(-X_{14}(ac), X_{23}(b)) \\ &= sign((1,3,2,4))\epsilon_{\theta((1,3,2,4))}(\overline{abc}) + sign((1,4,2,3))\epsilon_{\theta((1,4,2,3))}(\overline{acb}) \\ &= \epsilon_5(\overline{a(bc-cb)}) = 0. \end{aligned}$$

The other is: $x = X_{14}(a)$,$y = X_{43}(b)$, $z = X_{24}(c)$, and

$$\begin{aligned} J(x,y,z) &= -\psi(X_{13}(ab), X_{24}(c)) + \psi(-X_{23}(cb), X_{14}(a)) \\ &= -sign((1,3,2,4))\epsilon_{\theta((1,3,2,4))}(\overline{abc}) - sign((2,3,1,4))\epsilon_{\theta((2,3,1,4))}(\overline{acb})) \\ &= -\epsilon_5(\overline{a(bc-cb)}) = 0 \end{aligned}$$

as $sign((1,3,4,2)) = 1, sign((1,4,2,3)) = sign((2,3,1,4)) = -1$ and

$$\theta((1,3,4,2)) = \theta((1,4,2,3)) = \theta((2,3,1,4)) = 5$$

for any $a,b,c \in R$. The proof is completed. $\square$

Remark 4.5 In view of the proof, for $m = 5,6$, the m-th coordinate doesn't need modular $2R$. In this case, ψ has already become a (super) 2-cocycle.

Since

$$\mathcal{U} = span_K\{\psi(X_{ij}(a), X_{kl}(b)) | a,b \in R \text{ } and \text{ } i,j,k,l \text{ } are \text{ } distinct\}$$

and $deg(X_{ij}(a)) = deg(X_{kl}(b))$ for distinct $1 \le i,j,k,l \le 4$, we obtain a central extension of Lie superalgebra $\mathfrak{st}(2,2,R)$ satisfying that $\mathcal{U}$ is the even part of the kernel :

$$0 \to (\mathcal{U})_{\bar{0}} \oplus (0)_{\bar{1}} \to \widehat{\mathfrak{st}}(2,2,R) \xrightarrow{\pi} \mathfrak{st}(2,2,R) \to 0, \tag{4.6}$$

i.e.

$$\widehat{\mathfrak{st}}(2,2,R) = ((\mathcal{U})_{\bar{0}} \oplus (0)_{\bar{1}}) \oplus \mathfrak{st}(2,2,R). \tag{4.7}$$

$(\widehat{\mathfrak{st}}(2,2,R),\pi)$ is a central extension of $\mathfrak{st}(2,2,R)$. It is similar to the $\mathfrak{st}(3,1,R)$ case, we define a Lie superalgebra $\mathfrak{st}(2,2,R)^\sharp$ to be the Lie superalgebra generated by the symbols $X^\sharp_{ij}(a)$, $a \in R$ and the K-linear space $\mathcal{U}$, with $\deg(X^\sharp_{ij}(a)) = \omega(i) + \omega(j)$ and $\deg(u) = \bar{0}$ for any $u \in \mathcal{U}$, satisfying the following relations:

$$a \mapsto X_{ij}{}^\sharp(a) \text{ is a } K\text{-linear mapping}, \tag{4.8}$$

$$[X^\sharp_{ij}(a), X^\sharp_{jk}(b)] = X^\sharp_{ik}(ab), \text{ for distinct } i,j,k, \tag{4.9}$$

$$[X^\sharp_{ij}(a), \mathcal{U}] = 0, \text{ for distinct } i,j, \tag{4.10}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ij}(b)] = 0, \text{ for distinct } i,j, \tag{4.11}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ik}(b)] = 0, \text{ for distinct } i,j,k, \tag{4.12}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kj}(b)] = 0, \text{ for distinct } i,j,k, \tag{4.13}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kl}(b)] = sign((i,j,k,l))\epsilon_{\theta((i,j,k,l))}(\overline{ab}), \text{ for distinct } j,k,i,l, \tag{4.14}$$

where $a, b \in R$,$1 \leq i,j,k,l \leq 4$. As $1 \in R$, $\mathfrak{st}(2,2,R)^\sharp$ is perfect. Clearly, there is a unique Lie superalgebra homomorphism $\rho : \mathfrak{st}(2,2,R)^\sharp \to \widehat{\mathfrak{st}}(2,2,R)$ such that $\rho(X^\sharp_{ij}(a)) = X_{ij}(a)$ and $\rho|_{\mathcal{U}} = id$.

As was done in Lemma 3.14, we have

Lemma 4.15 $\rho : \mathfrak{st}(2,2,R)^\sharp \to \widehat{\mathfrak{st}}(2,2,R)$ *is a Lie superalgebra isomorphism.*

Now we can state the main theorem of this section.

Theorem 4.16 $(\widehat{\mathfrak{st}}(2,2,R),\pi)$ *is the universal central extension of* $\mathfrak{st}(2,2,R)$ *and hence*

$$H_2(\mathfrak{st}(2,2,R)) \cong (\mathcal{U})_{\bar{0}} \oplus (0)_{\bar{1}}.$$

Proof: Suppose that

$$0 \to \mathcal{V} \to \widetilde{\mathfrak{st}}(2,2,R) \xrightarrow{\tau} \mathfrak{st}(2,2,R) \to 0$$

is a central extension of $\mathfrak{st}(2,2,R)$. We must show that there exists a Lie algebra homomorphism $\eta : \widehat{\mathfrak{st}}(2,2,R) \to \widetilde{\mathfrak{st}}(2,2,R)$ so that $\tau \circ \eta = \pi$. Thus, by Lemma 4.15, it suffices to show that there exists a Lie algebra homomorphism $\xi : \mathfrak{st}(2,2,R)^\sharp \to \widetilde{\mathfrak{st}}(2,2,R)$ so that $\tau \circ \xi = \pi \circ \rho$.

We choose an appropriate preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ under τ, and check them satisfying (4.8)-(4.14). The difference from the proof of Theorem 3.18 is to treat $[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)]$, which is also denoted by $\nu_{kl}^{ij}(a,b)$.

We first have

$$\nu_{kj}^{il}(bc, a) = (-1)^{(\omega(k)+\omega(l))(\omega(k)+\omega(j))}\nu_{kl}^{ij}(ba, c).$$

Then taking $b = 1$ or $c = 1$,

$$\nu_{kj}^{il}(b, a) = (-1)^{(\omega(k)+\omega(l))(\omega(k)+\omega(j))}\nu_{kl}^{ij}(a, b) = (-1)^{(\omega(k)+\omega(l))(\omega(k)+\omega(j))}\nu_{kl}^{ij}(ba, 1) \tag{4.17}$$

where $a, b \in R$ and i, j, k, l are distinct.

For $1 \leq m \leq 4$, there exists an element $(i, j, k, l) \in P_m$, such that $\omega(i) + \omega(j) = \bar{0}$, by (3.24), we obtain

$$2\nu_{kl}^{ij}(a, b) = 0 \tag{4.18}$$

where $a, b \in R$. As in the proof of Theorem 3.18, one has

$$\nu_{kl}^{ij}(\mathcal{I}_2, 1) = 0. \tag{4.19}$$

By (4.16), the equation holds for any $(i, j, k, l) \in \bigsqcup_{m=1}^{4} P_m$.

On the other hand, if $m = 5, 6$, for all $(i, j, k, l) \in P_m$, $\omega(i) + \omega(j) = \bar{1}$, then

$$\begin{aligned}
\nu_{kl}^{ij}(c(ab - ba), 1) &= \nu_{kl}^{ij}(ab - ba, c) = \nu_{kl}^{ij}(ab + (-1)^{\omega(i)+\omega(j)}ba, c) \\
&= [\widetilde{X}_{ij}(ab + (-1)^{\omega(i)+\omega(j)}ba), \widetilde{X}_{kl}(c)] \\
&= \Big[[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ij}(1)], \widetilde{X}_{kl}(c)\Big] \\
&= 0
\end{aligned}$$

for $a, b, c, \in R$, which shows

$$\nu_{kl}^{ij}(\mathcal{I}_0, 1) = 0 \tag{4.20}$$

for $(i, j, k, l) \in P_5 \bigsqcup P_6$.

The rest of the proof is similar to Theorem 3.18, we can obtain $\xi : \mathfrak{st}(2,2,R)^\sharp \to \widetilde{\mathfrak{st}}(2,2,R)$. The only difference is that we need paying attention to the sign of the restriction of ξ on $\mathcal{U}$ as the 5-th and 6-th coordinate component of $\mathcal{U}$ is R_0. Let $\xi(\epsilon_m(\bar{a})) = sign((i,j,k,l))\nu_{kl}^{ij}(1,a)$, where $sign(i,j,k,l)$ is defined before Lemma 4.3. It is easy to see that the choice of sign coincides with the (super) skew-symmetry and (4.17). Thus, the Lie homomorphism ψ is well defined on $\mathcal{U}$. $\square$

Remark 4.20 Note that $H_2(\mathfrak{st}(2,2,R)) \cong R_2^4 \oplus R_0^2$. Even 2 is an invertible element of K so that $R = 2R$ and $R_2 = 0$, R_0 is not necessarily equal to 0. Particularly, if R is commutative, then $\mathcal{I}_0 = R[RR] = 0$ and $R_0 = R$. In this case, $H_2(\mathfrak{st}(2,2,R)) \cong R^2$ which is not trivial.

§5 Concluding remarks

Combining Theorem 1.19, Theorem 2.1, Theorem 3.18 and Theorem 4.15, we completely determined $H_2(\mathfrak{st}(m,n,R))$ for $m+n \geq 3$.

Theorem 5.1 *let K be a unital commutative ring and R be a unital associative K-algebra. Assume that R has a K-basis containing the identity element. Then*

$$H_2(\mathfrak{st}(m,n,R)) = \begin{cases} 0 & \text{for } m+n=3 \text{ and } m+n \geq 5 \\ (0)_{\bar{0}} \oplus (R_2^6)_{\bar{1}} & \text{for } m=3, n=1 \\ (R_2^4 \oplus R_0^2)_{\bar{0}} \oplus (0)_{\bar{1}} & \text{for } m=2, n=2 \end{cases}$$

which are $\mathbb{Z}_2$-graded spaces.

It then follows from [MP] that

Theorem 5.2 *let K be a unital commutative ring and R be a unital associative K-algebra. Assume that R has a K-basis containing the identity element. Then*

$$H_2(sl_n(R)) = \begin{cases} (HC_1(R))_{\bar{0}} \oplus (0)_{\bar{1}} & \text{for } m+n=3 \text{ and } m+n \geq 5 \\ (HC_1(R))_{\bar{0}} \oplus (R_2^6)_{\bar{1}} & \text{for } m=3, n=1 \\ (R_2^4 \oplus R_0^2 \oplus HC_1(R))_{\bar{0}} \oplus (0)_{\bar{1}} & \text{for } m=2, n=2 \end{cases}$$

where $HC_1(R)$ is the first cyclic homology group of the associative K-algebra R (See [L]).

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